CALCULATION OF ONE-LOOP VERTEX CORRECTIONS IN QUANTUM FIELD THEORY

A Thesis

Presented to the Honors Program of

Angelo State University

In Partial Fulfillment of the

Requirements for Highest University Honors

BACHELOR OF SCIENCE

by

CHARLIE PRESTON BREWER II

May 2019

Major: Mathematics and Physics

CALCULATION OF ONE-LOOP VERTEX CORRECTIONS

IN QUANTUM FIELD THEORY

by

CHARLIE PRESTON BREWER II

APPROVED:

Dr. David Bixler, Chair

Dr. Eddie F. Holik III

Dr. John Smith III

April 23, 2019

APPROVED:

Dr. Shirley M. Eoff May 3, 2019
Director of the Honors Program

Acknowledgements

I would like to thank my advisor, Dr. David Bixler, for allowing me to conduct research with him the past two years. Our research has greatly helped me to prepare for graduate school. Additionally, his humor helped make the many hours of incredibly frustrating calculations almost bearable. I would also like to thank the other members of my thesis committee, Dr. Trey Holik and Dr. Trey Smith, for agreeing to read and review my thesis. More so, I would like to thank them and the other math and physics professors for helping me in my journey to learn, and making it enjoyable along the way. I would like to thank Dr. Shirley Eoff for giving me the opportunity to complete the Honors Program and this thesis, both of which have been among the most fulfilling parts of my time at ASU. Of course, I would also like to thank my friends and family for their support throughout my life, but especially throughout the stressful time of completing this thesis.

Abstract

The greatest conceptual advancements of theoretical physics in the early 1900s were the development of quantum mechanics and special relativity. Special relativity tells us that measurements are relative, that they depend on the relative motion of the observers. For instance, an observer that is traveling very quickly (a significant fraction of the speed of light) will experience time more slowly than an observer at rest. Quantum mechanics tells us that fundamental particles do not behave exactly or deterministically; they exhibit wavelike behavior such as self interference under the double slit experiment. Despite how well each of these theories has been confirmed in their appropriate contexts, their mathematical frameworks are incompatible without some serious modifications. The theory developed to merge these two is quantum field theory. This combined theory, though significantly more complex, allows for predictions of much more fundamental phenomena such as the cross section of scattering processes in the relativistic limit and the calculation of the magnetic moment of the electron.

Contents

Chapter 1

Introduction

The early 1900s were a time of great change and advancement in theoretical physics. Many intuitive conceptions of the world that were held by physicists and non-physicists alike were found to be false. Even though this was one hundred years ago, physics from this time period came to be known as "modern" physics, as opposed to the physics before then which is known as "classical" physics. Classical physics includes topics such as classical mechanics and electrodynamics. Classical mechanics focuses on the behavior of systems that one might encounter in everyday life including combinations of ramps, spheres, springs, boxes, pulleys, pendulums, and rolling carts. Electrodynamics focuses on the behavior of charged objects in the presence of electric and magnetic fields. The revolutionary advancements that led to modern physics fall into two main categories: special relativity and quantum mechanics. Special relativity states that time, length, and position are not absolute: observers in different positions moving at different velocities will measure different values for these quantities. Quantum mechanics states that the behavior of the universe is never exactly knowable, especially when it comes to fundamental particles such as photons and electrons. Despite the fact that both of these concepts have been verified by countless

Physical Review D

experiments, they are not immediately compatible with each other. This fact led to the development of quantum field theory. However, before discussing QFT itself, it is helpful to first review the primary results of special relativity and quantum mechanics.

1.1 Special Relativity

Everything in Special Relativity is derived from two postulates, originally put forth by Albert Einstein[1]:

1. The laws of physics are the same in all inertial (non-accelerating) reference frames.

2. The speed of light in a vacuum is the same in all inertial reference frames.

These postulates, despite their simplicity, lead to some rather surprising relationships between space and time.

We consider a setup in which light is emitted straight upward where it is incident on a plane mirror, and then reflects directly back down to a detector at the same level as the source. If the distance from the light source to the mirror is L_0, then the total time light takes to reach the detector is $\Delta t_0 = 2L_0/c$ where c is the speed of light.

We now consider a situation where the setup is moving at velocity u to the right relative to the observer. Light is emitted, reaches the mirror at time $\Delta t/2$, and then reaches the detector at time Δt. In this case, the light travels distance $L = 2\sqrt{L_0^2 + (u\Delta t/2)^2}$. Thus

$$\Delta t = \frac{2L}{c} = \frac{2\sqrt{L_0^2 + (u\Delta t/2)^2}}{c}. \tag{1.1}$$

We find that

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - u^2/c^2}}. \tag{1.2}$$

This effect is known as time dilation: less time passes for a moving system as measured by a stationary observer than for a stationary system[1].

We now consider a similar setup where the source, mirror, and detector are moving at velocity u in the same direction as the light is emitted. Suppose the light takes time Δt_1 to reach the mirror and time Δt_2 to return to the detector. Then the distance travelled from the source to the mirror is $c\Delta t_1 = L + u\Delta t_1$ so $\Delta t_1 = L/(c-u)$ and the distance travelled back to the detector is $c\Delta t_2 = L - u\Delta t_2$ so $\Delta t_2 = L/(c+u)$. Thus, the total time taken is

$$\begin{aligned} \Delta t &= \Delta t_1 + \Delta t_2 \\ &= \frac{L}{c-u} + \frac{L}{c+u} \\ &= \frac{2Lc}{(c^2-u^2)} = \frac{2L}{c}\frac{1}{1-u^2/c^2}. \end{aligned} \tag{1.3}$$

But, we already found that $\Delta t = \frac{\Delta t_0}{\sqrt{1-u^2/v^2}} = \frac{2L_0}{c}\frac{1}{\sqrt{1-u^2/c^2}}$. Setting the two expressions for Δt equal to each other, we find $L = L_0\sqrt{1-u^2/c^2}$. This effect is known as length contraction: lengths in a moving system as measured by a stationary observer are shorter than those measured for a stationary system.

In general, we would like to find a set of rules for converting measurements from one stationary system O to another system O' that is moving at velocity u with respect to O. Without loss of generality, we choose O and O' to have their axes oriented in the same direction and that O' has velocity u in the x-direction. The rules for converting between these two reference frames are called Lorentz transformations and they are presented here without proof. If an event is observed at (x, y, z, t) in frame O, then the

same event will be observed at (x', y', z', t') in frame O' where

$$\begin{aligned} x' &= \frac{x - ut}{\sqrt{1 - u^2/c^2}} \\ y' &= y \\ z' &= z \\ t' &= \frac{t - (u/c^2)x}{\sqrt{1 - u^2/c^2}}. \end{aligned} \tag{1.4}$$

We also obtain a different relation for total energy, momentum, and mass given by

$$E^2 = (pc)^2 + (mc^2)^2. \tag{1.5}$$

If the particle is at rest, then $p = 0$ and we obtain the more famous equation

$$E = mc^2. \tag{1.6}$$

For convenience, hereafter we choose units such that $c = 1$.

1.2 Quantum Mechanics

Quantum mechanics is derived from two principles[2]. The first is the principle of superposition which states that a microscopic system exists as a superposition or combination of different possible states. The other principle is the principle of indeterminacy which says that measuring a microscopic system causes it to collapse into only one of the different component states. Further, there is no way to exactly predict into which state it will collapse; we can only predict the probability for observing each possible state. For example, an electron can be measured to be either spin up or spin down. At any given time, the electron will exist as some combination, or superposition,

of spin up and spin down. Then, suppose the electron is measured to be spin up. The act of measuring the electron causes it to collapse into a state that is entirely spin up. As time passes, the electron returns to a mixed state of spin up and spin down.

Formally, for any observable quantity O, the possible outcomes of the measurement are given by a set of vectors $|A_1\rangle$, $|A_2\rangle$, $|A_3\rangle$,... (there may be a finite or an infinite number of possible outcomes). These are called eigenvectors of the observable O. These vectors are equipped with an inner product, an operation that takes two vectors and returns a number. For two vectors $|A\rangle$ and $|B\rangle$, we denote the inner product by $\langle B|A\rangle$. If $|A\rangle$ and $|B\rangle$ are real-valued then $\langle A|B\rangle = \langle B|A\rangle$. In general, $\langle A|B\rangle = \langle B|A\rangle^*$, where $*$ denotes the complex conjugate. A vector is said to be normalized if $\langle A|A\rangle = 1$. For convenience, we assume any state vector in quantum mechanics is normalized so that the magnitude is equal to one. At any given time, the quantum system exists as a vector $|A\rangle$ that is a combination of these eigenvectors,

$$|A\rangle = a_1 |A_1\rangle + a_2 |A_2\rangle + a_3 |A_3\rangle \dots \tag{1.7}$$

If the system is measured, it will be measured to be in only one of the states $|A_i\rangle$. The probability of the state being observed as $|A_i\rangle$ is given by $|a_i|^2$.

A common observable of interest is position. Although it is not what one normally thinks of as a vector, the vector used to represent the position state of a system is a function called the wavefunction $\Psi(x,t)$. In general, Ψ is complex-valued. The magnitude squared of Ψ gives the probability density for a particle to be observed at a particular location x at time t. That is, if $P(x,t)$ is the probability for a particle to be observed at location x at time t, then

$$P(x,t) = |\Psi(x,t)|^2 = \Psi^*(x,t)\Psi(x,t). \tag{1.8}$$

For wavefunctions, the inner product is defined by

$$\langle \Psi_2 | \Psi_1 \rangle = \int_{-\infty}^{\infty} \Psi_2^*(x,t)\Psi_1(x,t)dx. \tag{1.9}$$

Thus, the normalization requirement says that $\int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)dx = 1$ at all times t. Physically this corresponds to the fact that the particle has to have total probability 1 of existing somewhere in the universe. To solve for a wavefunction, we use the Schrodinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V(x)\Psi = i\frac{\partial \Psi}{\partial t}, \tag{1.10}$$

where $V(x)$ is a function describing the potential energy. Similar to choosing $c = 1$, we also choose units such that $\hbar = 1$. This combined choice of units is known as *natural units*. In natural units, time and distance have the same units and energy, momentum, and mass all have the same units.

Chapter 2

Klein-Gordon and Dirac Equations

2.1 Klein-Gordon Equation

One of the first attempts at merging quantum mechanics and special relativity was the development of the Klein-Gordon equation. One reason that the Schrodinger equation cannot be compatible with special relativity is because special relativity treats space and time equally whereas the Schrodinger equation is second order in space (involves the second derivative) but only first order in time. An obvious possible solution is to construct a wave equation that is second order in both space and time. Using the energy-momentum relation

$$E^2 = p^2 + m^2, \tag{2.1}$$

and that in quantum mechanics, the energy operator is $E = i\frac{\partial}{\partial t}$ and the momentum operator is $p = -i\frac{\partial}{\partial x}$, we obtain the equation

$$-\frac{\partial^2 \Psi}{\partial t^2} = -\frac{\partial^2 \Psi}{\partial x^2} + m^2 \Psi \tag{2.2}$$

or the more common form

$$\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial^2\Psi}{\partial x^2} + m^2\Psi = 0. \tag{2.3}$$

However, this equation allows for negative values for probability density and thus, Ψ cannot be interpreted as the probability density. Nonetheless, this equation will prove useful at a later point.

2.2 Dirac Equation

The cause for negative probability densities in the Klein-Gordon equation was the fact that it is second order in time. Because of this, Paul Dirac sought a way to make the equation first order in time while retaining its relativistic invariance. He thought to do this by essentially factoring the equation. However, to do this he realized he would need a set of anticommuting matrices so that the cross terms would vanish. This set of four 4×4 matrices are the ubiquitous Dirac matrices and the resulting equation is the Dirac equation,

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{2.4}$$

Since this is now a matrix equation, ψ is no longer a scalar function, but a four-component column vector called a spinor. Two components of this spinor have solutions that have positive energy, but the other two components have negative energy. This perplexed physicists at the time and they considered simply disregarding these solutions much as one might disregard a negative solution for time in a kinematics problem. However, Dirac suggested otherwise. He suggested that these are instead particles that have the opposite charge of ordinary matter particles. An apparently extraneous solution led to the prediction of antimatter long before it was actually discovered[3].

2.3 Quantizing the Klein-Gordon Field

In quantum field theory, instead of considering the wave function corresponding to distinct particles, we instead treat each type of matter as its own field. These fields are coupled to the other fields so that they can affect each other just as matter, of course, interacts with other matter. The general process is to set up the initial configuration of the field, see how it evolves, and observe the results. To do this, we often need to find the amplitude for an excitation in the field to propagate from one point at one time to some other point at some other time. Let $\psi(x)$ be the operator that creates a particle of the ψ field at x, where x is the four-vector (t, x, y, z); a more thorough treatment of four-vectors is given in [4]. This operator must act on the vacuum, or ground state, $|0\rangle$. Thus our desired state is $\psi(x)\,|0\rangle$. The probability that a particle propagates from x to y is thus represented by $\langle 0|\psi(y)\psi(x)|0\rangle$. Hence, we need to determine how to evaluate this expression.

Recall that we can write a field, or in this case a field operator, in terms of its Fourier transform. That is,

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}). \tag{2.5}$$

In momentum space, the spatial derivatives of the Klein-Gordon equation simply become factors of $-i|\mathbf{p}|$, so the equation becomes

$$\frac{\partial^2 \phi}{\partial t^2} + (|\mathbf{p}|^2 + m^2)\phi = 0. \tag{2.6}$$

This is the equation of motion for a simple harmonic oscillator with frequency $\omega = \sqrt{|\mathbf{p}|^2 + m^2}$. Recall that the Hamiltonian, or total energy, for a classical simple

harmonic oscillator is given by

$$H = \frac{p^2}{2m} + \frac{1}{2}kx^2. \tag{2.7}$$

Choosing $m = 1$ for convenience and replacing the position x with the field operator $\phi(x)$, we obtain $H = \frac{p^2}{2} + \frac{1}{2}\omega^2\phi^2$, where $\omega = \frac{k}{m}$. Further, recall from nonrelativistic quantum mechanics that we define the raising and lowering operators by $a^\dagger = \sqrt{\frac{\omega}{2}}(\phi + \frac{i}{\omega}p)$ and $a = \sqrt{\frac{\omega}{2}}(\phi - \frac{i}{\omega}p)$, respectively. Thus, we find $\phi = \frac{1}{\sqrt{2\omega}}(a + a^\dagger)$ and $p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$. The canonical commutation relation $[x, p] = i$ is then equivalent to $[a, a^\dagger] = 1$. We may then rewrite the Hamiltonian as

$$H = \omega(a^\dagger a + \frac{1}{2}). \tag{2.8}$$

If we define the vacuum state $|0\rangle$ such that it is "destroyed" by the lowering operator so that $a|0\rangle = 0$, we see that $|0\rangle$ is an eigenstate of H with eigenvalue $\frac{1}{2}\omega$ since

$$H|0\rangle = \omega(a^\dagger a + \frac{1}{2})|0\rangle = \omega a^\dagger a|0\rangle + \frac{1}{2}\omega|0\rangle = \frac{1}{2}\omega|0\rangle. \tag{2.9}$$

An apparently empty state still has energy, called the zero-point energy. This anomalous energy does not have a clear significance and, since only energy differences matter in general, it does not affect the result of predictions. A nonempty state with n particles, represented by $|n\rangle = (a^\dagger)^n|0\rangle$, is an eigenstate of H with energy $(n + \frac{1}{2})\omega$. For example, choose $n = 2$. Then, using the commutation relation and the definition of the

vacuum state, we find

$$\begin{aligned} H\left|2\right\rangle &= \omega(a^\dagger a + \frac{1}{2})a^\dagger a^\dagger \left|0\right\rangle \\ &= \omega a^\dagger a a^\dagger a^\dagger \left|0\right\rangle + \frac{1}{2}\omega a^\dagger a^\dagger \left|0\right\rangle \\ &= \omega a^\dagger (a^\dagger a + 1)a^\dagger + \frac{1}{2}\omega a^\dagger a^\dagger \left|0\right\rangle \\ &= \omega a^\dagger a^\dagger a a^\dagger \left|0\right\rangle \omega a^\dagger a^\dagger \left|0\right\rangle + \frac{1}{2}\omega a^\dagger a^\dagger \left|0\right\rangle \\ &= \omega a^\dagger a^\dagger (a^\dagger a + a)\left|0\right\rangle + \omega a^\dagger a^\dagger \left|0\right\rangle + \frac{1}{2}\omega a^\dagger a^\dagger \left|0\right\rangle \\ &= \omega a^\dagger a^\dagger \left|0\right\rangle + \omega a^\dagger a^\dagger \left|0\right\rangle + \frac{1}{2}\omega a^\dagger a^\dagger \left|0\right\rangle \\ &= \omega(2 + \frac{1}{2})\left|2\right\rangle . \end{aligned} \tag{2.10}$$

For the Fourier transform of the Klein-Gordon field operator $\phi(\mathbf{x})$, we treat each value of momentum in the integral as its own oscillator, with independent creation and annihilation operators $a^\dagger_{\mathbf{p}}$ and $a_{\mathbf{p}}$ so that

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \tag{2.11}$$

and

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a^\dagger_{\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{x}}\right) . \tag{2.12}$$

Letting $\mathbf{p} \to -\mathbf{p}$ in the second terms and using $\omega_{\mathbf{p}} = \omega_{-\mathbf{p}}$, we may also write these as

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right) e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{2.13}$$

The conjugate momentum operator is defined similarly,

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right) e^{i\mathbf{p}\cdot\mathbf{x}}. \tag{2.14}$$

With these adjustments, we define $[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}'}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')$. The delta function is included so that when $\mathbf{p} \neq \mathbf{p}'$, they commute and are independent, and when $\mathbf{p} = \mathbf{p}'$, the commutator is 1 after integration. The factor of $(2\pi)^3$ is essentially to cancel the $\frac{1}{(2\pi)^3}$ from the definition of the Fourier transform. Then $[\phi(\mathbf{x}), \pi(\mathbf{x}')]$ works out correctly,

$$[\phi(\mathbf{x}), \pi(\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}'). \tag{2.15}$$

We now seek to write the Hamiltonian of the Klein-Gordon field in terms of the field operator $\phi(\mathbf{x})$ and conjugate momentum operator $\pi(\mathbf{x})$. The Klein-Gordon Hamiltonian [5] is given by

$$H = \int d^3x \mathcal{H} = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]. \tag{2.16}$$

We rewrite each piece in terms of creation and annihilation operators. For convenience, we will do so for each term individually. For the first term,

$$\begin{aligned}
\frac{1}{2}\pi^2 &= \frac{1}{2}\pi\pi \\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3} e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}(-i)^2\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{2}(a_{\mathbf{p}} - a^\dagger_{\mathbf{p}})(a_{\mathbf{p}'} - a^\dagger_{\mathbf{p}'}) \\
&= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3} e^{i(\mathbf{p}+\mathbf{p}')\cdot x}\frac{-\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{4}(a_{\mathbf{p}} - a^\dagger_{\mathbf{p}})(a_{\mathbf{p}'} - a^\dagger_{\mathbf{p}'}).
\end{aligned} \tag{2.17}$$

For the second term,

$$\begin{aligned}
\frac{1}{2}(\nabla\phi)^2 &= \frac{1}{2}\nabla\phi\cdot\nabla\phi \\
&= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3} e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}\frac{-\mathbf{p}\cdot\mathbf{p}'}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}(a_{\mathbf{p}} + a^\dagger_{\mathbf{p}})(a_{\mathbf{p}'} + a^\dagger_{\mathbf{p}'}).
\end{aligned} \tag{2.18}$$

Finally, the last term becomes,

$$\begin{aligned}\frac{1}{2}m^2\phi^2 &= \frac{1}{2}m^2\phi\phi \\ &= \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3}e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}\frac{m^2}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}(a_{\mathbf{p}}+a^\dagger_{\mathbf{p}})(a_{\mathbf{p}'}+a^\dagger_{\mathbf{p}'}).\end{aligned} \tag{2.19}$$

Putting the pieces together yields

$$\begin{aligned}H = \int d^3x \int \frac{d^3p}{(2\pi)^3}\int \frac{d^3p'}{(2\pi)^3}e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}}\Bigg[&\frac{-\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{4}(a_{\mathbf{p}}-a^\dagger_{\mathbf{p}})(a_{\mathbf{p}'}-a^\dagger_{\mathbf{p}'}) \\ &+\frac{-\mathbf{p}\cdot\mathbf{p}'}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}(a_{\mathbf{p}}+a^\dagger_{\mathbf{p}})(a_{\mathbf{p}'}+a^\dagger_{\mathbf{p}'})\Bigg].\end{aligned} \tag{2.20}$$

We use the spatial integral so that $\int d^3x e^{i(\mathbf{p}+\mathbf{p}')\cdot\mathbf{x}} \to \delta(\mathbf{p}+\mathbf{p}')$ which we then use to evaluate the p' integral. This enforces $\mathbf{p}' = -\mathbf{p}$ in the integrand. We also exploit the fact that since we integrate over all values of $\mathbf{p}$, we may let $\mathbf{p} \to -\mathbf{p}$ in any given term. We then find

$$\begin{aligned}H &= \int \frac{d^3p}{(2\pi)^3}\frac{\omega_{\mathbf{p}}}{4}\left[-(a_{\mathbf{p}}-a^\dagger_{\mathbf{p}})(a_{\mathbf{p}'}-a^\dagger_{\mathbf{p}'})+(a_{\mathbf{p}}-a^\dagger_{\mathbf{p}})(a_{\mathbf{p}'}-a^\dagger_{\mathbf{p}'})\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{\omega_{\mathbf{p}}}{4}\Big[-a_{\mathbf{p}}a_{-\mathbf{p}}+a_{\mathbf{p}}a^\dagger_{\mathbf{p}}-a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}}-a^\dagger_{-\mathbf{p}}a^\dagger_{\mathbf{p}}+a_{\mathbf{p}}a^\dagger_{-\mathbf{p}}+a_{\mathbf{p}}a^\dagger_{\mathbf{p}} \\ &\qquad\qquad +a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}}+a^\dagger_{-\mathbf{p}}a_{\mathbf{p}}\Big] \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{\omega_{\mathbf{p}}}{2}\left[a_{\mathbf{p}}a^\dagger_{\mathbf{p}}+a^\dagger_{-\mathbf{p}}a_{-\mathbf{p}}\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left[\frac{1}{2}a_{\mathbf{p}}a^\dagger_{\mathbf{p}}+\frac{1}{2}a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left[a^\dagger_{\mathbf{p}}a_{\mathbf{p}}+\frac{1}{2}a_{\mathbf{p}}a^\dagger_{\mathbf{p}}-\frac{1}{2}a^\dagger_{\mathbf{p}}a_{\mathbf{p}}\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left[a^\dagger_{\mathbf{p}}a_{\mathbf{p}}+\frac{1}{2}[a_{\mathbf{p}},a^\dagger_{\mathbf{p}}]\right] \\ &= \int \frac{d^3p}{(2\pi)^3}\omega_{\mathbf{p}}\left[a^\dagger_{\mathbf{p}}a_{\mathbf{p}}+\delta(0)\right].\end{aligned} \tag{2.21}$$

The second term is essentially an infinite constant. It is the sum of the zero point

energy of each oscillator, one for each value of $\mathbf{p}$. We again appeal to the fact that only energy differences matter in calculations and omit this term. We also use $E_{\mathbf{p}}$ in place of $\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$ since this is the correct relativistic energy for a particle with momentum. Hence, we write

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}}. \tag{2.22}$$

Notice that $a^\dagger_{\mathbf{p}}$ and $a^\dagger_{\mathbf{q}}$ commute for all $\mathbf{p}$ and $\mathbf{q}$ and thus, $a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} |0\rangle = a^\dagger_{\mathbf{q}} a^\dagger_{\mathbf{p}} |0\rangle$, which means that particles obeying the Klein-Gordon equation are bosons.

We now digress briefly to address the choice of normalization of states. We define $|0\rangle$ so that $\langle 0|0\rangle = 1$. For a state $|\mathbf{p}\rangle$, it might be natural to try and define $\langle \mathbf{q}|\mathbf{p}\rangle = (2\pi)^3 \delta(\mathbf{p} - \mathbf{q})$, where delta function is necessary since if $\mathbf{p} \neq \mathbf{q}$, then $\langle \mathbf{q}|\mathbf{p}\rangle = \langle 0|a_{\mathbf{q}} a^\dagger_{\mathbf{p}}|0\rangle = \langle 0|a^\dagger_{\mathbf{p}} a_{\mathbf{q}}|0\rangle = 0$. The problem with defining normalization in this way is that it is not Lorentz-invariant. Consider a boost in 3-direction so that $\mathbf{p} \to \mathbf{p}'$ and $\mathbf{q} \to \mathbf{q}'$. Using a delta function identity,

$$\begin{aligned} \delta^{(3)}(\mathbf{p} - \mathbf{q}) &= \delta^{(3)}(\mathbf{p}' - \mathbf{q}') \frac{dp'_3}{dp_3} \\ &= \delta^{(3)}(\mathbf{p}' - \mathbf{q}') \frac{E'}{E}. \end{aligned} \tag{2.23}$$

We see that replacing $\delta^{(3)}(\mathbf{p} - \mathbf{q})$ with $E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q})$ is Lorentz-invariant. We therefore define $|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} a^\dagger_{\mathbf{p}} |0\rangle$ so that

$$\langle \mathbf{q}|\mathbf{p}\rangle = 2E_{\mathbf{p}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}). \tag{2.24}$$

We now return to the problem of finding the amplitude for a particle of the Klein-Gordon field to propagate from a spacetime point x to a point y, $\langle 0|\phi(y)\phi(x)|0\rangle$. It turns out that this quantity does not obey causality, in that it does not vanish if y is

outside the light-cone of x. That is, it does not vanish if $|\mathbf{x}-\mathbf{y}| > |x^0 - y^0|$. We instead consider the commutator $[\phi(x), \phi(y)]$, which in nonrelativistic quantum mechanics, tells whether one observation affects another observation. Thus, one would hope that $[\phi(x), \phi(y)]$ vanishes if y is outside the light-cone of x, which it does. We define $D_F(x-y) = \langle 0|[\phi(x), \phi(y)]|0\rangle$ to be the propagator for a particle of the Klein-Gordon field. More properly, this is known as the Feynman propagator for the Klein-Gordon field since it will later be used in Feynman diagrams, originally developed by Richard Feynman. This quantity is also a Green's function for the Klein-Gordon equation, meaning that it is a solution to

$$(\partial^2 + m^2)D_F(x-y) = -i\delta^{(4)}(x-y). \tag{2.25}$$

With this, we can solve for D_F by taking the Fourier transform. If

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \widetilde{D}_F(p), \tag{2.26}$$

the equation becomes

$$(-p^2 + m^2)\widetilde{D}_F(p) = -i. \tag{2.27}$$

We solve and take the inverse transform which gives

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}. \tag{2.28}$$

The p^0 integral has poles when

$$(p^0)^2 - |\mathbf{p^2}| + m^2 = 0, \tag{2.29}$$

or equivalently, when

$$p^0 = \pm\sqrt{|\mathbf{p}|^2 + m^2} = \pm E_{\mathbf{p}}. \tag{2.30}$$

Since we want to avoid these singularities, we shift the $-E_{\mathbf{p}}$ singularity above the axis and shift the $+E_{\mathbf{p}}$ singularity below the axis by adding $i\epsilon$ to the denominator. We must also decide how to close this contour. If $x^0 > y^0$, then we close the contour below (where $p^0 = a - bi$) so that the semicircle is damped by a factor of $e^{-b(x-y)}$ and thus vanishes as we let the radius of the semicircle go to infinity (since b gets large). If instead, $y^0 > x^0$, we close the contour above, where $p^0 = a + bi$, so that the semicircle is damped by a factor of $e^{-b(y-x)}$, and still vanishes. In closing the contour below, we pick up the residue at $+E_{\mathbf{p}}$ which is $\frac{1}{2E_{\mathbf{p}}}$. Closing the contour above, we pick up the residue at $-E_{\mathbf{p}}$ which is $\frac{-1}{2E_{\mathbf{p}}}$. Hence, if $x^0 > y^0$, we obtain

$$\begin{aligned} D_F(x-y) &= \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(x-y)} \\ &= \langle 0|\phi(x)\phi(y)|0\rangle\,. \end{aligned} \tag{2.31}$$

If $x^0 < y^0$, we get the same thing but with x and y interchanged. Hence,

$$D_F(x-y) = \langle 0|T\phi(x)\phi(y)|0\rangle\,, \tag{2.32}$$

where we have defined the time ordering operator T to place the operators in order, with the latest to the left.

2.4 Quantizing the Dirac Field

Although we gave the Dirac equation in (2.4), we did not discuss the gamma matrices that appear in the equation. The gamma matrices are a set of four matrices, defined by the property that they anticommute with each other, but each one squares

to $\pm I$, where I is the four by four identity matrix. That is,

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I. \tag{2.33}$$

There are many possible representations of the Dirac matrices, the two most notable of which are the Dirac representation and the Weyl representation, but the specific representation is often unimportant. Explicitly, in the Weyl representation, the gamma matrices are

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} & \gamma^1 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ \gamma^2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} & \gamma^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{2.34}$$

Recall the Pauli matrices, defined by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.35}$$

Then, defining $\sigma^\mu = (I_2, \sigma^1, \sigma^2, \sigma^3)$ and $\overline{\sigma}^\mu = (I_2, -\sigma^1, -\sigma^2, -\sigma^3)$, we can write the gamma matrices as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}. \tag{2.36}$$

We now discuss solutions to the Dirac equation. Let ξ^1 and ξ^2 be orthonormal two-component spinors. Also, let $\sqrt{p \cdot \sigma}$ denote the positive square root of the eigenvalues of $p \cdot \sigma$ and similarly for $\sqrt{p \cdot \overline{\sigma}}$. Then the Dirac equation has solutions of the form

$$\psi(x) = u^s(p) e^{-ip \cdot x}, \tag{2.37}$$

where $p^2 = m^2$ and

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\, \xi^s \\ \sqrt{p \cdot \overline{\sigma}}\, \xi^s \end{pmatrix}. \tag{2.38}$$

There is another set of solutions, given by

$$\psi(x) = v^s(p)e^{+ip\cdot x}. \tag{2.39}$$

Here,

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\overline{\sigma}}\eta^s, \end{pmatrix} \tag{2.40}$$

where η^1 and η^2 are another set of orthonormal spinors. It can be shown that

$$u^\dagger(p)u(p) = 2E_{\mathbf{p}}, \tag{2.41}$$

which is not Lorentz invariant. To remedy this, we define $\overline{u}(p) = u^\dagger(p)\gamma^0$, so that

$$\overline{u}^r(p)u^s(p) = 2m\delta^{rs}, \tag{2.42}$$

which is Lorentz invariant. Similarly

$$\overline{v}^r(p)v^s(p) = -2m\delta^{rs}. \tag{2.43}$$

Further, u and v are orthogonal in that

$$\overline{u}^r(p)v^s(p) = \overline{v}^r(p)u^s(p) = 0. \tag{2.44}$$

In evaluating Feynman diagrams, we will often need to sum over the different spin states. Thus, a useful identity is

$$\sum_{s=1,2} u^s(p)\overline{u}^s(p) = \gamma_\mu p^\mu + m = \not{p} + m, \tag{2.45}$$

where we have introduced the Feynman slash notation $p_\mu \gamma^\mu = \not{p}$. Similarly,

$$\sum_{s=1,2} v^s(p)\overline{v}^s(p) = \not{p} - m. \tag{2.46}$$

Since ψ is a four componenet spinor, there are four creation operators, $a^{s\dagger}_{\mathbf{p}}$ which correspond to creating fermions (like electrons) and $b^{s\dagger}_{\mathbf{p}}$ which corresponds to creating antifermions (like positrons), and the associated annihilation operators. In analogy to (2.11), we define

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a^s_{\mathbf{p}} u^s(pe^{-ip\cdot x}) + b^{s\dagger}_{\mathbf{p}} v^s(p) e^{+ip\cdot x} \right). \tag{2.47}$$

Thus, conjugating both sides and multiplying by γ^0,

$$\overline{\psi}(x) = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(b^s_{\mathbf{p}} \overline{v}^s(p) e^{-ip\cdot x} + a^{s\dagger}_{\mathbf{p}} \overline{u}^s(p) e^{+ip\cdot x} \right). \tag{2.48}$$

Since ψ describes a fermion, we expect it to be antisymmetric under the interchange of operators. We thus postulate the *anticommuation* relations

$$\{a^r_{\mathbf{p}}, a^{s\dagger}_{\mathbf{q}}\} = \{b^r_{\mathbf{p}}, b^{s\dagger}_{\mathbf{q}}\} = \delta^{(3)}(\mathbf{p} - \mathbf{q})\delta^{rs}, \tag{2.49}$$

and all others to be zero. With this definition, it can be shown that

$$\begin{aligned} \{\psi_a(\mathbf{x}), \psi_b(\mathbf{y})\} &= \delta^{(3)}(\mathbf{x} - \mathbf{y})\delta_{ab}; \\ \{\psi_a(\mathbf{x}), \psi_b(\mathbf{y})\} &= \{\psi^\dagger_a(\mathbf{x}), \psi^\dagger_b \mathbf{y}\} = 0, \end{aligned} \tag{2.50}$$

where a and b refer to the different components of ψ. Similar to (2.19), the Hamiltonian can be written

$$H = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} \sum_s \left(a^{s\dagger}_{\mathbf{p}} a^s_{\mathbf{p}} + b^{s\dagger}_{\mathbf{p}} b^s_{\mathbf{p}} \right). \tag{2.51}$$

The vacuum state is defined to be destroyed by the annihilation operators,

$$a^s_{\mathbf{p}} |0\rangle = b^s_{\mathbf{p}} |0\rangle = 0. \tag{2.52}$$

One-particle states are defined as

$$|\mathbf{p}, s\rangle = \sqrt{2E_{\mathbf{p}}} a^{s\dagger}_{\mathbf{p}} |0\rangle \tag{2.53}$$

so that the inner product

$$\langle \mathbf{p}, r | \mathbf{q}, s\rangle = 2E_{\mathbf{p}}(2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})\delta^{rs} \tag{2.54}$$

is Lorentz invariant.

We seek to find a propagator for the Dirac field. As before, we do so by finding a Green's function for the Dirac equation by setting

$$(i\partial\!\!\!/ - m)S_F(x - y) = i\delta^{(4)}(x - y), \tag{2.55}$$

where $\partial\!\!\!/ = \gamma^\mu \partial_\mu$. Then, taking the Fourier transform gives

$$(p\!\!\!/ - m)\widetilde{S}_F(p) = i \tag{2.56}$$

and thus,

$$\widetilde{S}_F(p) = \frac{i}{p\!\!\!/ - m} = \frac{i(p\!\!\!/ + m)}{p^2 - m^2}. \tag{2.57}$$

Inverting the transform and shifting the denominator by $i\epsilon$ gives

$$\begin{aligned} S_F(x - y) &= \int \frac{d^4p}{(2\pi)^4} \frac{i(p\!\!\!/ + m)}{p^2 - m^2 + i\epsilon} e^{-ip\cdot(d-y)} \\ &= \langle 0 | T\psi(x)\overline{\psi}(y)|0\rangle \,, \end{aligned} \tag{2.58}$$

where we have defined the time ordering operator T to pick up a minus sign when the fermion operators are interchanged.

Chapter 3

Feynman Diagrams

3.1 Perturbation Theory

There are some problems in physics that we know very well and for which we are able to find solutions. Perhaps the prime example of this is the simple harmonic oscillator, a problem that is encountered in many different areas of physics and one for which we have an exact solution. Often, however, we encounter problems that are very complicated and can only be solved approximately. In these cases, a useful approach is to take a well known problem and "perturb" it by making a slight modification to the problem setup or equation so that it more closely resembles the complicated problem. This is the idea behind what is called perturbation theory. Note that perturbation theory is not a field in physics like classical mechanics or electrodynamics; it is simply a method that can be applied to a problem in any field of physics.

As we have seen in field theory, a given Hamiltonian will determine how a system evolves over time, or how the field operators will behave. The Hamiltonian of the Standard Model of Particle Physics, which describes all known interactions of all known particles, is very long and unwieldy. As such, we will not attempt to describe this system. Instead we will take the simplest system we know, the free or noninteracting

system developed in the previous chapter, and we will add an interaction term, or perturbation, to allow for particle interactions that exhibit more interesting behavior.

3.2 Correlation Functions

In order to be able to calculate scattering cross sections, we must first learn how to calculate propagation amplitudes in an interacting theory. That is, expressions of the form

$$\langle \Omega | \phi(x)\phi(y) | \Omega \rangle \, . \tag{3.1}$$

First, notice that the field operator $\phi(x)$ that we use here is not the same as the one used in the previous chapters since we are using a different Hamiltonian and the Hamiltonian determines the time dependence of the field operators. Additionally, notice that the ground state $|\Omega\rangle$ is not necessarily the same as the ground state from the free theory, the vacuum state $|0\rangle$. One reason why we must consider a different ground state $|\Omega\rangle$ is that in an interacting theory, the ground state does not necessarily have zero energy.

We begin by examining the behavior of the field operators in the interacting theory. Our new Hamiltonian is given by

$$H = H_0 + H_{\text{int}} = H_{KG} + \int d^3x \frac{\lambda}{4!}\phi^4(\mathbf{x}), \tag{3.2}$$

where H_{KG} is the Klein-Gordon Hamiltonian as in (2.21). The theory developed using this Hamiltonian is known as "ϕ^4 theory". Although this will not be the Hamiltonian we actually use in developing QED, it is worth examining as it shows the process and principles for quantizing theories with other Hamiltonians.

With this modification to the Hamiltonian, we must find how to express $\phi(x)$ and $|\Omega\rangle$ in terms of the free field operator $\phi_I(x)$ and the vacuum state $|0\rangle$, where we now

denote the free field operator as $\phi_I(x)$ since the time dependence makes it an operator in the interaction picture. The interaction picture is a formulation of quantum mechanics where both operators and state vectors carry time dependence.. Since $\phi_I(x)$ is defined using only H_0 by

$$\phi_I(x) = e^{iH_0(t-t_0)}\phi(t_0, \mathbf{x})e^{-iH_0(t-t_0)}, \tag{3.3}$$

we may write $\phi(x)$ in terms of $\phi_I(x)$ by

$$\begin{aligned}\phi(x) &= e^{iH(t-t_0)}e^{-iH_0(t-t_0)}\phi_I(x)e^{iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &= U^\dagger(t,t_0)\phi_I(x)U(t,t_0),\end{aligned} \tag{3.4}$$

where we have defined the time evolution operator $U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$. It is easily seen that $U(t,t_0)$ is unitary, meaning $UU^\dagger$ is the identity operator. It can also be shown that $U(t,t_0)$ also satisfies a differential equation, the Schrodinger equation. To show this, we first must write the interaction term of the Hamiltonian H_{int} in the interaction picture by including time dependence. Thus, we define

$$\begin{aligned}H_I(t) &= e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)} \\ &= \int d^3x \frac{\lambda}{4!}e^{iH_0(t-t_0)}\phi^4(\mathbf{x})e^{-iH_0(t-t_0)} \\ &= \int d^3x \frac{\lambda}{4!}e^{iH_0(t-t_0)}\phi(\mathbf{x})\left(e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}\right)\phi(\mathbf{x})\left(e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}\right) \\ &\qquad\qquad \times \phi(\mathbf{x})\left(e^{-iH_0(t-t_0)}e^{iH_0(t-t_0)}\right)\phi(\mathbf{x})e^{-iH_0(t-t_0)} \\ &= \int d^3x \frac{\lambda}{4!}\phi_I^4(\mathbf{x}).\end{aligned} \tag{3.5}$$

Then,

$$\begin{aligned} i\frac{\partial}{\partial t}U(t,t_0) &= i\left(\frac{\partial}{\partial t}e^{iH_0(t-t_0)}\right)e^{-iH(t-t_0)} + ie^{iH_0(t-t_0)}\left(\frac{\partial}{\partial t}e^{-iH(t-t_0)}\right) \\ &= -H_0e^{iH_0(t-t_0)}e^{-iH(t-t_0)} + e^{iH_0(t-t_0)}He^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}\left(-H_0\right)e^{-iH(t-t_0)} + e^{iH_0(t-t_0)}\left(H\right)e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}\left(H-H_0\right)e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH(t-t_0)} \\ &= e^{iH_0(t-t_0)}H_{\text{int}}e^{-iH_0(t-t_0)}e^{H_0(t-t_0)}e^{-iH(t-t_0)} \\ &= H_I(t)U(t,t_0), \end{aligned} \tag{3.6}$$

where we have used the fact that $H_{\text{int}} = H - H_0$ and also that H_0 and $e^{iH_0(t-t_0)}$ commute. Since these are operators, we do not simply obtain e^{-iH_It}, as one might expect from a differential equation of this form. Instead, the solution is

$$\begin{aligned} U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_2}dt_2 H_I(t_1)H_I(t_2) \\ + (-i)^3\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\int_{t_0}^{t_2}dt_3 H_I(t_1)H_I(t_2)H_I(t_3) + \ldots, \end{aligned} \tag{3.7}$$

where T is the time ordering operator defined in (2.32). This can be verified by using the Leibniz rule and differentiating each term, which gives the previous term times $-iH_I$, as desired. This also obeys the usual condition for the time evolution operator which is $U(t,t_0) = 1$ when $t = t_0$. Without showing the details, we may use an identity to alter the limits of integration to obtain

$$\begin{aligned} U(t,t_0) &= 1 + (-i)\int_{t_0}^{t} dt_1 H_I(t_1) + \frac{(-i)^2}{2!}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2 T\{H_I(t_1)H_I(t_2)\} + \cdots \\ &= T\left\{\exp\left[-i\int_{t_0}^{t'}dt' H_I(t')\right]\right\}. \end{aligned} \tag{3.8}$$

The operator U also satisfies the properties

$$U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3) \tag{3.9}$$

and

$$U^\dagger(t_1, t_2) = U(t_2, t_1). \tag{3.10}$$

We now seek to find an expression for $|\Omega\rangle$ in terms of operators whose properties we already know. We essentially evolve the vacuum state through time 0 to time T,

$$|0\rangle \to e^{-iH(t-t_0)} |0\rangle = e^{-iH(T-0)} |0\rangle = e^{-iHT} |0\rangle . \tag{3.11}$$

Since energy is an observable quantity, its eigenstates form a complete basis so we may expand in terms of energy eigenstates,

$$e^{-iHT} |0\rangle = \sum_{n\geq 0} e^{-iHT} |n\rangle \langle n|0\rangle = \sum_{n\geq 0} e^{-iE_nT} |n\rangle \langle n|0\rangle . \tag{3.12}$$

Naturally, the ground state is defined as the state with least energy. That is, the energy eigenstate for $n = 0$. Then,

$$e^{-iHT} |0\rangle = e^{-iE_0T} |\Omega\rangle \langle \Omega|0\rangle + \sum_{n\geq 1} e^{-iE_nT} |n\rangle \langle n|0\rangle . \tag{3.13}$$

To isolate the $|\Omega\rangle$ term, we need to dispose of the other terms. To do so, let $T \to \infty(1 - i\epsilon)$ for some small positive constant ϵ. Then every term in (3.13) gets damped by a factor of the form $\lim_{T\to\infty} e^{-E_nT\epsilon}$. Since $E_n > E_0$ for $n > 0$, the damping effect is greater for each of these terms and these terms may be neglected. Thus,

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0T} \langle \Omega|0\rangle\right)^{-1} e^{-iHT} |0\rangle . \tag{3.14}$$

Since we are taking a limit as T goes to infinity, we may shift it by a constant t_0. Doing so, and using the fact that $H_0 |0\rangle = 0$ so that $e^{-iH_0 T} |0\rangle = (1) |0\rangle$:

$$\begin{aligned}
|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(T+t_0)} \langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)} |0\rangle \\
&= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0-(-T))} \langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0-(-T))} |0\rangle \\
&= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0-(-T))} \langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0-(T))} e^{-iH_0(-T-t_0)} |0\rangle \\
&= \lim_{T\to\infty(1-i\epsilon)} \left(e^{-iE_0(t_0-(-T))} \langle\Omega|0\rangle\right)^{-1} U(t_0,-T) |0\rangle ,
\end{aligned} \tag{3.15}$$

where we also used the definition of U from (3.4). Hence, ignoring the constant factor in front, $|\Omega\rangle$ is just $|0\rangle$ evolved through time using the time evolution operator $U(t_0, -T)$. A similar process yields

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \langle 0| U(T,t_0) \left(e^{-iE_0(T-t_0)} \langle 0|\Omega\rangle\right)^{-1} . \tag{3.16}$$

We now have all the pieces to evaluate $\langle\Omega|\phi(x)\phi(y)|\Omega\rangle$. If $x_0 > y_0 > t_0$,

$$\begin{aligned}
\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} &\langle 0| U(T,t_0) \left(e^{-iE_0(T-t_0)} \langle 0|\Omega\rangle\right)^{-1} \left\{U^\dagger(x^0,t_0)\phi_I(x)U(x^0,t_0)\right\} \\
&\times \left\{U^\dagger(y^0,t_0)\phi_I(y)U(y^0,t_0)\right\} \left(e^{-iE_0(t_0-(-T))} \langle\Omega|0\rangle\right)^{-1} U(t_0,-T) |0\rangle \\
= \lim_{T\to\infty(1-i\epsilon)} &\left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} \\
&\times \langle 0| U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T) |0\rangle .
\end{aligned} \tag{3.17}$$

This expression would look simpler without the factor in front. To eliminate it, we use (3.15) and (3.16) to divide by 1 in the form

$$1 = \langle\Omega|\Omega\rangle = \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}\right)^{-1} \langle 0|U(T,t_0)U(t_0,-T)|0\rangle . \tag{3.18}$$

Also, each operator in the numerator is in time order, so we may further simplify the expression by inserting a time ordering operator:

$$\begin{aligned}
\langle\Omega|\phi(x)\phi(y)|\Omega\rangle &= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\, U(T,x^0)\phi_I(x)U(x^0,y^0)\phi_I(y)U(y^0,-T)\,|0\rangle}{\langle 0|U(T,-T)|0\rangle} \\
&= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\, T\left\{\phi_I(x)\phi_I(y)U(T,x^0)U(x^0,y^0)U(y^0,-T)\right\}|0\rangle}{\langle 0|U(T,-T)|0\rangle} \\
&= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\, T\left\{\phi_I(x)\phi_I(y)U(T,-T)\right\}|0\rangle}{\langle 0|U(T,-T)|0\rangle} \\
&= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\, T\left\{\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}|0\rangle}{\langle 0|T\left\{\exp\left[-i\int_{-T}^{T}dtH_I(t)\right]\right\}|0\rangle}.
\end{aligned} \tag{3.19}$$

This is the final form we will need.

3.3 Wick's Theorem

We have thus reduced the problem to evaluating expressions of the form $\langle 0|T\left\{\phi_I(x)\phi_I(y)\right\}|0\rangle$. Now, to evaluate this expression, we will need to develop what is known as Wick's theorem. First, we write $\phi_I(x)$ in terms of positive and negative frequency terms, $\phi^+(x)$ and $\phi^-(x)$, where $\phi^+(x)$ is defined by

$$\phi_I^+(x) = \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot x} a_{\mathbf{p}} \tag{3.20}$$

and $\phi^-(x)$ is defined by

$$\phi_I^-(x) = \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} a_{\mathbf{p}}^\dagger \tag{3.21}$$

so that

$$\phi_I(x) = \phi_I^+(x) + \phi_I^-(x). \tag{3.22}$$

Then, if $x^0 > y^0$,

$$\begin{aligned}
\langle 0|T\{\phi(x)\phi(y)\}|0\rangle &= \langle 0|\phi_I(x)\phi_I(y)|0\rangle \\
&= \langle 0|\left(\phi^+(x)+\phi^-(x)\right)\left(\phi^+(y)+\phi^-(y)\right)|0\rangle \\
&= \langle 0|\,\phi^+(x)\phi^+(y)+\phi^+(x)\phi^-(y)+\phi^-(x)\phi^+(y)+\phi^-(x)\phi^-(y)\,|0\rangle \\
&= \langle 0|\,\phi^+(x)\phi^-(y)\,|0\rangle \\
&= \langle 0|\,\phi^-(y)\phi^+(x)+\left[\phi^+(x),\phi^-(y)\right]|0\rangle \\
&= \left[\phi^+(x),\phi^-(y)\right],
\end{aligned} \tag{3.23}$$

where any term with all $a_{\mathbf{p}}$ operators to the right of the $a^\dagger_{\mathbf{p}}$ operators becomes zero, since $a_{\mathbf{p}}|0\rangle = 0$ and $\langle 0|\,a^\dagger_{\mathbf{p}} = 0$. Such a term is said to be in *normal order*. We define the normal ordering operator N to place its argument in normal order. For example,

$$N(a_{\mathbf{p}}a^\dagger_{\mathbf{q}}a^\dagger_{\mathbf{l}}a_{\mathbf{q}}) = a^\dagger_{\mathbf{q}}a^\dagger_{\mathbf{l}}a_{\mathbf{p}}a_{\mathbf{q}}. \tag{3.24}$$

If instead $y^0 > x^0$, we would have obtained $[\phi^+(y),\phi^-(x)]$. Hence, we define the *Wick contraction* of two operators by

$$\overbrace{\phi(x)\phi}(y) = \begin{cases} \left[\phi^+(x),\phi^-(y)\right], & x^0 > y^0 \\ \left[\phi^+(y),\phi^-(x)\right], & y^0 > x^0. \end{cases} \tag{3.25}$$

This is the same as the Feynman propagator from (2.32),

$$\overbrace{\phi(x)\phi}(y) = D_F(x-y). \tag{3.26}$$

We see that

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y)+\overbrace{\phi(x)\phi}(y)\}. \tag{3.27}$$

Note that in this case, with only two field operators, it does not matter if we place the contraction inside or outside the normal ordering operator since the contraction is just a number. We seek to generalize this result to any number of fields, leading to Wick's theorem.

Theorem. (*Wick's theorem*)
Suppose, without loss of generality, that $x_1^0 \geq x_2^0 \geq \cdots \geq x_m^0$ and let $\phi_i = \phi(x_i)$. Then

$$T\{\phi_1\phi_2\cdots\phi_m\} = N\{\phi_1\phi_2\cdots\phi_m + \text{all possible contractions}\}, \tag{3.28}$$

where "all possible contractions" means there is a term for each possible way of contracting the fields in pairs and, in any such term, not all fields are necessarily contracted with some other field.

Proof. We proceed by induction on m. We have already shown the base case where $m = 2$. Now, assume this holds for $m-1$. Since the fields are already time ordered,

$$\begin{aligned} T\{\phi_1\phi_2\cdots\phi_m\} &= \phi_1 T\{\phi_2\cdots\phi_m\} \\ &= \phi_1 N\{\phi_2\cdots\phi_m + (\text{contractions involving } \phi_2\cdots\phi_m)\} \\ &= (\phi_1^+ + \phi_1^-) N\{\phi_2\cdots\phi_m + (\text{contractions involving } \phi_2\cdots\phi_m)\}. \end{aligned} \tag{3.29}$$

The ϕ_1^- term is on the left, and thus, already in normal order. Hence, we need only move the ϕ_1^+ to the right by commuting it past every other term. Further, since ϕ_1^+ commutes with all other ϕ_i^+, moving ϕ_1^+ past ϕ_i only gives an extra term of $[\phi_1^+, \phi_i^-]$. That is,

$$\begin{aligned} \phi_1^+\phi_i &= \phi_1^+(\phi_i^+ + \phi_i^-) \\ &= \phi_1^+\phi_i^+ + \phi_1^+\phi_i^- \\ &= \phi_i^+\phi_1^+ + \phi_i^-\phi_1^+ + [\phi_1^+, \phi_i^-] \\ &= \phi_i\phi_1^+ + \overbrace{\phi_1\phi_i}. \end{aligned} \tag{3.30}$$

Then, for the term without any contractions, to move ϕ_1^+ to the right, we must commute with every ϕ_i. Hence,

$$\phi_1^+\phi_2\cdots\phi_m = \phi_2\cdots\phi_m\phi_1^+ + \overbrace{\phi_1\phi_2}\cdots\phi_m + \cdots + \overbrace{\phi_1\phi_2\cdots\phi_m}. \tag{3.31}$$

The other terms of (3.29) are not affected by having some operators already contracted, since the contraction becomes a number. In this case, ϕ_1^+ must commute past each operator that is not already contracted. Therefore,

$$T\{\phi_1\phi_2\cdots\phi_m\} = N\{\phi_1\phi_2\cdots\phi_m + \text{all possible contractions}\}, \tag{3.32}$$

as desired.

■

For example,

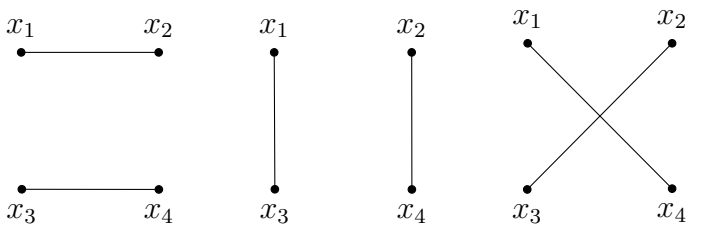
$$\begin{aligned} T\{\phi_1\phi_2\phi_3\phi_4\} = N\{&\phi_1\phi_2\phi_3\phi_4 + \overbrace{\phi_1\phi_2}\phi_3\phi_4 + \overbrace{\phi_1\phi_2\phi_3}\phi_4 + \overbrace{\phi_1\phi_2\phi_3\phi_4} + \phi_1\overbrace{\phi_2\phi_3}\phi_4 \\ &+ \phi_1\overbrace{\phi_2\phi_3\phi_4} + \phi_1\phi_2\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\overbrace{\phi_2\phi_3}\phi_4} + \overbrace{\phi_1\overbrace{\phi_2\phi_3}\phi_4}\}. \end{aligned} \tag{3.33}$$

Any term that contains uncontracted operators will vanish when taking the expectation value, since such a term contains a field operator that annihilates the vacuum since it is placed in normal order. Thus, the only terms that are non-zero are the fully contracted

ones. Then

$$\begin{aligned}\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle &= \langle 0|\,\overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\overbrace{\phi_2\phi_3}\phi_4} + \overbrace{\phi_1\overbrace{\phi_2\phi_3\phi_4}}\,|0\rangle \\ &= \langle 0|\,D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) \\ &\qquad + D_F(x_1-x_4)D_F(x_2-x_3)\,|0\rangle \\ &= D_F(x_1-x_2)D_F(x_3-x_4) + D_F(x_1-x_3)D_F(x_2-x_4) \\ &\qquad + D_F(x_1-x_4)D_F(x_2-x_3).\end{aligned} \tag{3.34}$$

We can interpret this expression as the different ways to connect four points. That is, we can represent $D_F(x_1-x_2)D_F(x_3-x_4)$, $D_F(x_1-x_3)D_F(x_2-x_4)$, and $D_F(x_1-x_4)D_F(x_2-x_3)$ by the following diagrams, respectively:



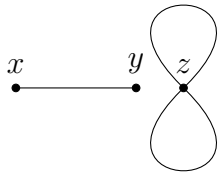
. (3.35)

This introduces the idea of *Feynman diagrams*, where we take points in spacetime and connect them in pairs, where the line in between represents a propagator which gives the amplitude for the particle to propagate from one point to the other. The total amplitude for a process is the sum of the different possible full contractions, each of which corresponds to a particular Feynman diagram. Considering the series expansion of (3.19), taking only the leading term gives the noninteracting result. The second term is

$$T\left\{\phi(x)\phi(y)\frac{-i\lambda}{4!}\int dt\int d^3z\phi^4(z)\right\}. \tag{3.36}$$

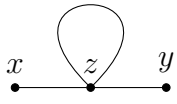
There are 15 possible ways to contract these fields in pairs, but some of these have the

same expression and diagram. One diagram is



(3.37)

obtained from a contraction of the form $\overbrace{\phi(x)\phi}(y)\int d^4z\overbrace{\phi(z)\phi}(z)\overbrace{\phi(z)\phi}(z)$. There are three contractions that lead to this same diagram, obtained from the three ways of contracting the $\phi(z)$ operators with each other. The other diagram is

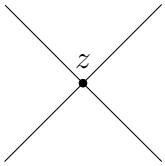


, (3.38)

obtained from a contraction of the form $\overbrace{\phi(x)\overbrace{\phi(y)\int d^4z\phi(z)}\phi}(z)\overbrace{\phi(z)\phi}(z)$. There are 12 different contractions that give this diagram, obtained from the four ways to contract $\phi(x)$ with a $\phi(z)$ and the three ways to contract $\phi(y)$ with one of the remaining $\phi(z)$. Then, for ϕ^4 theory, we get the following *position space Feynman rules* for Feynman diagrams:

1. For each propagator (internal line), 

$= D_F(x-y)$;

2. For each vertex, 

$= (-i\lambda)\int d^4z$;

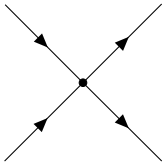
3. For each external point, 

$= 1$.

A more common approach is to express a diagram in terms of the momentum associated with each internal line. This is essentially done by taking the Fourier

transform of each factor associated with the position space Feynman rules. Then we have the *momentum space Feynman rules*:

1. For each propagator, $\xrightarrow{p} = \frac{i}{p^2 - m^2 + i\epsilon}$;

2. For each vertex,  $= -i\lambda$;

3. For each external point, $\overset{x}{\bullet}\longleftarrow_{p} = e^{-ip\cdot x}$;

4. Impose momentum conservation at each vertex;

5. Integrate over each undetermined loop momentum.

A similar process gives the Feynman rules for the Dirac field operators. Recall that time ordering for fermions includes a minus sign when the fields are interchanged. If $y^0 > x^0$, then $T\{\psi(x)\psi(y)\} = -\psi(y)\psi(x)$. Then, contractions for the Dirac field are given by

$$\overbrace{\psi(x)\overline{\psi}}(y) = S_F(x-y), \quad \overbrace{\psi(x)\psi}(y) = \overline{\psi}(x)\overline{\psi}(y) = 0. \tag{3.39}$$

Wick's theorem is the same as before since all necessary minus signs are accounted for in the definition of T and N.

Toward calculating cross sections, we first introduce the invariant matrix $\mathcal{M}$ defined by

$$i\mathcal{M} = \begin{pmatrix} \text{sum of fully connected,} \\ \text{amputated Feynman diagrams} \end{pmatrix}. \tag{3.40}$$

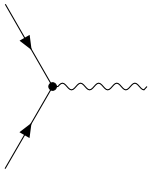
By amputation, we mean to remove any loops that start and end on the same external leg. The reason we may neglect any diagram with a disconnected piece and consider only the fully connected diagrams is that diagrams that are not fully connected cancel from the numerator and denominator of (3.19).

We now need a formula to relate the amplitude calculated from Feynman diagrams to the macroscopic cross section for a process to occur. In the cases we will consider, there will be two incoming particles and two outgoing particles. The incoming particles will both be of the same mass and the outgoing particles will both be of the same mass. If the incoming particles have four-momenta p_A and p_B and energies E_A and E_B, and the outgoing particles have four-momenta p_1 and p_2, all in the center-of-mass frame, then the formula we seek is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{CM}} |\mathcal{M}(p_A, p_B \to p_1, p_2)|^2, \tag{3.41}$$

where $v_A = \frac{p_{A,z}}{E_A}$ and similarly for p_B. Here, E_{cm} is the total energy of the system in the center-of-mass frame and $d\Omega$ is the differential solid angle element, $d\Omega = \sin(\theta) d\theta d\phi$.The derivation of this formula can be found here [5].

Finally, to calculate diagrams in QED, we need a different Hamiltonian which gives a different set of Feynman rules. For the Hamiltonian, we include a term for the fermion field, as before, but we replace the scalar particle ϕ with a vector particle A_μ, where each component of A_μ satisfies the Klein-Gordon equation. This vector particle corresponds to the photon. We also include the interaction Hamiltonian $H_{\text{int}} = \int d^3x e\overline{\psi}\gamma^\mu\psi A_\mu$. This interaction term allows for the fermions (electrons and positrons) to interact with photons. With these adjustments, we get the following modifications to the Feynman rules:

1. Vertex:  $= -ie\gamma^\mu$;

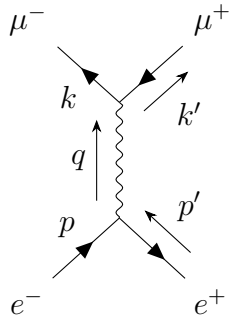
2. Photon propagator: $\xrightarrow{\quad q \quad}$ $= \dfrac{-ig_{\mu\nu}}{q^2 + i\epsilon}$.

Chapter 4

QED Processes

4.1 e^+e^- to $\mu^+\mu^-$

The first diagram in QED we calculate is electron positron annihilation into a muon antimuon pair. The diagram and associated expression are



$$= \bar{v}(p')(-ie\gamma^\mu)u(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k)(-ie\gamma^\nu)v(k'). \tag{4.1}$$

Then

$$i\mathcal{M} = \frac{ie^2}{q^4}\left(\overline{v}(p')\gamma^\mu u(p)\right)\left(\overline{u}(k)\gamma_\mu v(k')\right), \tag{4.2}$$

where we used $g_{\mu\nu}$ so that $\gamma^\nu \to \gamma_\mu$. To compute the differential cross section, we need $|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^*$. Since $(\overline{v}\gamma^\mu u)^* = \overline{u}\gamma^\mu v$, we see that

$$\begin{aligned}
|\mathcal{M}|^2 &= \mathcal{M}\mathcal{M}^* \\
&= \frac{e^4}{q^4}\Big[\Big(\overline{v}(p')\gamma^\mu u(p)\Big)\Big(\overline{u}(k)\gamma_\mu v(k')\Big)\Big]\Big[\Big(\overline{v}(p')\gamma^\nu u(p)\Big)\Big(\overline{u}(k)\gamma_\nu v(k')\Big)\Big]^* \\
&= \frac{e^4}{q^4}\Big[\Big(\overline{v}(p')\gamma^\mu u(p)\Big)\Big(\overline{u}(k)\gamma_\mu v(k')\Big)\Big]\Big[\Big(\overline{u}(p)\gamma^\nu v(p')\Big)\Big(\overline{v}(k')\gamma_\nu u(k)\Big)\Big] \\
&= \frac{e^4}{q^4}\Big(\overline{v}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu v(p')\Big)\Big(\overline{u}(k)\gamma_\mu v(k')\overline{v}(k')\gamma_\nu u(k)\Big),
\end{aligned} \tag{4.3}$$

where $q^4 = (q \cdot q)^2$ and where we used a different index for the gamma matrices in $\mathcal{M}^*$ so that the notation is valid. We also used the fact that $\overline{u}\gamma^\mu v$ and any similar expressions is a scalar and may be moved freely throughout the equation. Since we want to compute the differential cross section, we need to find $|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^*$. There are two facts that can help simplify this expression. The first is that electron and positron beams are often unpolarized, so the beams will contain about half of each possible spin. That is, the beams will be about half spin-up and half spin-down. Thus, we simply average over the spin polarizations for the electron and positron. Further, muon detectors do not detect polarization—they detect muons of any spin. Thus, we sum over the polarizations of the muons. Hence, we seek to evaluate

$$\frac{1}{4}\sum_{s=1}^{2}\sum_{s'=1}^{2}\sum_{r=1}^{2}\sum_{r'=1}^{2}|\mathcal{M}(s,s' \to r,r')|^2. \tag{4.4}$$

The first part of the expression is

$$\overline{v}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu v(p'). \tag{4.5}$$

Recall the completeness relations from (2.45) and (2.46). To be able to use the completeness relations to simplify this expression, we would like to be able to move

$v(p')$ next to $\overline{v}(p')$. To do so, we switch to index notation for the matrices giving

$$\begin{aligned}\sum_s \sum_{s'} \overline{v}_a^{s'}(p')\gamma^\mu_{ab} u_b^s(p)\overline{u}_c^s(p)\gamma^\nu_{cd} v_d^{s'}(p') &= \sum_s \sum_{s'} v_d^{s'}(p')\overline{v}_a^{s'}(p')\gamma^\mu_{ab} u_b^s(p)\overline{u}_c^s(p)\gamma^\nu_{cd} \\ &= (\not{p}' - m)_{da}\gamma^\mu_{ab}(\not{p} + m)_{bc}\gamma^\nu_{cd} \\ &= \operatorname{tr}\Big((\not{p}' - m)\gamma^\mu(\not{p} + m)\gamma^\nu\Big).\end{aligned} \tag{4.6}$$

To show the last equality, suppose we wanted to find the trace of some matrix $D = ABC$. In index notation, this is written $A_{ij}B_{jk}C_{kl} = D_{il}$. Then, to find the trace of D, we simply sum over the diagonal components by setting $i = l$. Thus,

$$\operatorname{tr}(D) = D_{ii} = A_{ij}B_{jk}C_{ki}, \tag{4.7}$$

as seen in (4.6). Doing the same for the second half of (4.3) and plugging back in gives

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4q^4}\operatorname{tr}\Big((\not{p}' - m_e)\gamma^\mu(\not{p} + m_e)\gamma^\nu\Big)\operatorname{tr}\Big((\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\Big). \tag{4.8}$$

Some properties that will be useful in evaluating this expression are $\operatorname{tr}(\gamma^\mu) = 0$, $\operatorname{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$, $\operatorname{tr}(\not{p}\gamma^\mu) = 4p^\mu$, and $\operatorname{tr}(\not{p}\not{k}) = 4p \cdot k$. To show the first, we introduce $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, which anticommutes with each γ^μ. We also have that $(\gamma^5)^2 = I$, where I is the 4×4 identity matrix. Then

$$\begin{aligned}\operatorname{tr}(\gamma^\mu) &= \operatorname{tr}(\gamma^5\gamma^5\gamma^\mu) \\ &= \operatorname{tr}(\gamma^5\gamma^\mu\gamma^5) \\ &= \operatorname{tr}(-\gamma^5\gamma^5\gamma^\mu) \\ &= -\operatorname{tr}(\gamma^\mu) \\ &= 0,\end{aligned} \tag{4.9}$$

where the second line used the cyclic property of trace and the third line used anticommutation. To show the second property, notice

$$\begin{aligned} \operatorname{tr}(\gamma^\mu\gamma^\nu) &= \operatorname{tr}(2g^{\mu\nu}I - \gamma^\nu\gamma^\mu) \\ &= 2g^{\mu\nu}\operatorname{tr}(I) - \operatorname{tr}(\gamma^\nu\gamma^\mu) \\ &= 8g^{\mu\nu} - \operatorname{tr}(\gamma^\mu\gamma^\nu), \end{aligned} \tag{4.10}$$

where the last line used the cyclic property of trace. Then $2\operatorname{tr}(\gamma^\mu\gamma^\nu) = 8g^{\mu\nu}$, and hence, $\operatorname{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$. With this, the last two properties are simple to show. We see

$$\begin{aligned} \operatorname{tr}(\not{p}\gamma^\mu) &= \operatorname{tr}(p_\nu\gamma^\nu\gamma^\mu) \\ &= p_\nu\operatorname{tr}(\gamma^\mu\gamma^\nu) \\ &= 4p_\nu g^{\mu\nu} \\ &= 4p^\mu. \end{aligned} \tag{4.11}$$

Similarly,

$$\begin{aligned} \operatorname{tr}(\not{p}\not{k}) &= \operatorname{tr}(p_\mu\gamma^\nu k_\nu\gamma^\nu) \\ &= p_\mu k_\nu\operatorname{tr}(\gamma^\mu\gamma^\nu) \\ &= 4p_\mu k_\nu g^{\mu\nu} \\ &= 4p\cdot k. \end{aligned} \tag{4.12}$$

Since the electron mass is much smaller than the muon mass, we may neglect it and set $m_e = 0$. Then, evaluating the first trace gives

$$\begin{aligned}
\mathrm{tr}\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu\right] &= \mathrm{tr}\left[\not{p}'\gamma^\mu(2p^\nu - \gamma^\nu\not{p})\right] \\
&= 2p^\nu\,\mathrm{tr}\left[\not{p}'\gamma^\mu\right] - \mathrm{tr}\left[\not{p}'\gamma^\mu\gamma^\nu\not{p}\right] \\
&= 8p^\nu p'^\mu - \mathrm{tr}\left[\not{p}'(2g^{\mu\nu} - \gamma^\nu\gamma^\nu)\not{p}\right] \\
&= 8p^\nu p'^\mu - 2g^{\mu\nu}\,\mathrm{tr}\left[\not{p}'\not{p}\right] + \mathrm{tr}\left[\not{p}'\gamma^\nu\gamma^\mu\not{p}\right] \\
&= 8p^\nu p'^\mu - 8g^{\mu\nu}p'\cdot p + \mathrm{tr}\left[(2p'^\nu - \gamma^\nu\not{p}')\gamma^\mu\not{p}\right] \\
&= 8p^\nu p'^\mu - 8g^{\mu\nu}p'\cdot p + 2p'^\nu\,\mathrm{tr}\left[\gamma^\mu\not{p}\right] - \mathrm{tr}\left[\gamma^\nu\not{p}'\gamma^\mu\not{p}\right] \\
&= 8p^\nu p'^\mu - 8g^{\mu\nu}p'\cdot p + 8p'^\nu p^\mu + \mathrm{tr}\left[\gamma^\mu\not{p}\right] - \mathrm{tr}\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu\right],
\end{aligned} \tag{4.13}$$

so

$$\mathrm{tr}\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu\right] = 4(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}p'\cdot p). \tag{4.14}$$

A similar process for the second trace yields

$$\mathrm{tr}\left[(\not{k} + m_\mu)\gamma_\mu(\not{k}' - m_\mu)\gamma_\nu\right] = 4(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' + m_\mu^2)). \tag{4.15}$$

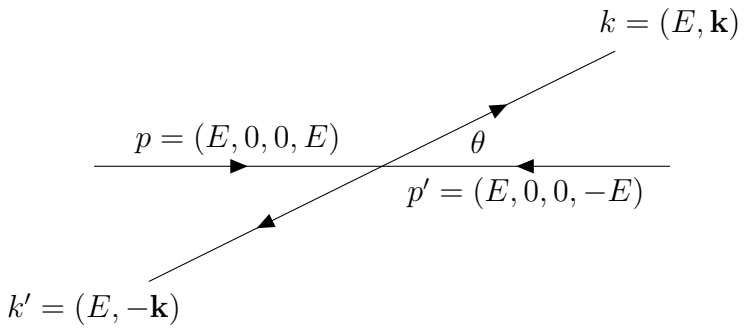
Note that the μ from m_μ is unrelated to the tensor index μ. It is simply to indicate that the mass is of the muon as opposed to the electron. Now, multiplying the traces together yields

$$\begin{aligned}
&\Big(p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}p'\cdot p\Big)\Big(k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' + m_\mu^2)\Big) \\
&\qquad = (p'\cdot k)(p\cdot k') + (p'\cdot k')(p\cdot k) - (p'\cdot p)(k\cdot k' + m_\mu^2) \\
&\qquad\qquad + (p'\cdot k')(p\cdot k) + (p'\cdot k)(p\cdot k') - (p\cdot p')(k\cdot k' + m_\mu^2) \\
&\qquad\qquad\quad - (p'\cdot p)(k'\cdot k) - (p'\cdot p)(k'\cdot k) + 4(p'\cdot p)(k\cdot k' + m_\mu^2) \\
&\qquad = 2(p'\cdot k)(p\cdot k') + 2m_\mu^2(p'\cdot p).
\end{aligned} \tag{4.16}$$

After restoring the factors of $16\frac{1}{4}\frac{e^4}{q^4}$, we finally obtain

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{q^4}\Big[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p'\cdot p)\Big]. \tag{4.17}$$

To be able to get an actual numerical value from this expression, we must pick a frame and work through the kinematics of the process. Naturally, we pick the center-of-mass frame. The diagram below shows the incoming and outgoing momenta of the particles, where we choose the z-axis to go left and right.



Using the energy-momentum relation $E^2 = |\mathbf{k}|^2 + m_\mu^2$, we see that $|\mathbf{k}| = \sqrt{E^2 - m_\mu^2}$. We will need the z component of $\mathbf{k}$, $\mathbf{k}\cdot\hat{z} = |\mathbf{k}|\cos(\theta)$. We also make use of the following relationships:

$$\begin{aligned}
p\cdot p' &= (E)(E) - 0 - 0 - (E)(-E) = 2E^2 \\
q^2 &= (p+p')^2 = p^2 + 2p\cdot p' + p'^2 = m_e^2 + 2(2E) + m_e^2 = 4E \\
p\cdot k &= p'\cdot k' = (E)(E) - 0 - 0 - (E)(|\mathbf{k}|\cos(\theta)) = E^2 - E|\mathbf{k}|\cos(\theta) \\
p\cdot k' &= p'\cdot k = (E)(E) - 0 - 0 - (E)(-|\mathbf{k}|\cos(\theta)) = E^2 + E|\mathbf{k}|\cos(\theta).
\end{aligned} \tag{4.18}$$

Plugging back in,

$$\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 &= \frac{8e^4}{(4E^2)^2}\Big[(E^2 - E|\mathbf{k}|\cos(\theta))^2 + (E^2 + E|\mathbf{k}|\cos(\theta))^2 + 2m_\mu^2 E^2\Big] \\
&= \frac{8}{16E^4}\Big[E^4 - 2E^3|k|\cos(\theta) + E^2|\mathbf{k}|^2\cos^2(\theta) + E^4 + 2E^3|\mathbf{k}|\cos(\theta) \\
&\qquad\qquad + E^2|\mathbf{k}|^2\cos^2(\theta) + 2m_\mu^2 E^2\Big] \\
&= \frac{e^4}{2E^4}\Big[2E^4 + 2E^2|\mathbf{k}|^2\cos^2(\theta) + 2m_\mu^2\Big] \\
&= \frac{e^4}{E^4}\Big[E^4 + m_\mu^2 + E^2(E^2 - m_\mu^2)\cos^2(\theta)\Big] \\
&= e^4\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2(\theta)\right].
\end{aligned} \tag{4.19}$$

Using the facts that $|v_A - v_B| = 2$ and $E = E_{CM}/2$, we can plug in for the formula for differential cross section,

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{2E_A 2E_B|v_A - v_B|}\frac{|\mathbf{k}|}{(2\pi)^2 4E_{cm}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 \\
&= \frac{1}{2E_{cm}^2}\frac{\sqrt{E^2 - m_\mu^2}}{16\pi^2\frac{E}{2}}\frac{1}{4}e^4\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2(\theta)\right] \\
&= \frac{e^4}{16\pi^2}\frac{1}{4E_{cm}^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2(\theta)\right] \\
&= \frac{\alpha^2}{4E_{cm}^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2(\theta)\right],
\end{aligned} \tag{4.20}$$

where we have also defined the *fine-structure constant*, $\alpha = \frac{e^2}{4\pi}$. To find the total cross section, we integrate over $d\Omega = \sin(\theta)d\theta d\phi$. For the first term in brackets, this simply

gives a factor of 4π. Integrating the second term,

$$\begin{aligned}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \cos^2(\theta)\sin(\theta) &= 2\pi \int_0^{\pi} d\theta \cos^2(\theta)\sin(\theta) \\ &= -2\pi \int_1^{-1} du \;\; u^2 \\ &= \frac{4\pi}{3}.\end{aligned} \tag{4.21}$$

Adding the terms in brackets,

$$\begin{aligned}4\pi\left(1+\frac{m_\mu^2}{E^2}\right)+\frac{4\pi}{3}\left(1-\frac{m_\mu^2}{E^2}\right) &= \frac{12\pi}{3}+\frac{4\pi}{3}+\frac{12\pi}{3}\frac{m_\mu^2}{E^2}-\frac{4\pi}{3}\frac{m_\mu^2}{E^2} \\ &= \frac{16\pi}{3}+\frac{8\pi}{3}\frac{m_\mu^2}{E^2} \\ &= \frac{16\pi}{3}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right).\end{aligned} \tag{4.22}$$

We now plug this in to finally obtain the total cross section,

$$\sigma = \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right). \tag{4.23}$$

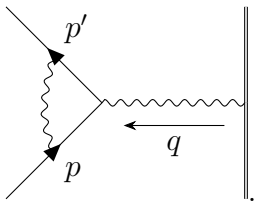
Chapter 5

One-Loop Corrections

5.1 Magnetic Moment

So far, we have only considered leading order diagrams: diagrams limited to two vertices and no loops. To be able to make accurate predictions in QED, however, we need to be able to evaluate diagrams to the next order. The problem with this is that in diagrams with a loop, we must integrate over all possible values of momentum for the loop, since it is not uniquely constrained. It is challenging to get an answer at all for these calculations, and even when we do, the answer is often divergent (QFT is notorious for divergences). Thus, we will need to employ a method to extract a meaningful, finite answer. The problem of infinities and the methods used to tame them have a rigorous treatment in *renormalization*, which is beyond the scope of this thesis.

The diagram of interest for this chapter will be the diagram corresponding to the scattering of an electron from a classical electric or magnetic field source:



(5.1)

To evaluate and derive physical meaning from this diagram, we introduce the electron vertex function Γ^μ, defined such that $-ie\Gamma^\mu$ represents the sum of all possible amputated interactions at the vertex. Naturally, this can depend on the incoming and outgoing momentum of the electron, so $\Gamma^\mu = \Gamma^\mu(p', p)$. Since the diagram with no loops is the only diagram where the vertex does not pick up extra factors of $\alpha = \frac{e^2}{4\pi}$ (it is the only diagram with only two vertices), we have that $\Gamma^\mu = \gamma^\mu$, to leading order. Further, since the diagram shown above is the only acceptable diagram where the interaction at the vertex contributes an extra factor of α, this diagram must be the only order α contribution to Γ^μ. Notice that Γ^μ is a vector; it carries a single index. Hence, it can only be a linear combination of p^μ, p'^μ, and γ^μ. Thus, we may write

$$\Gamma^\mu = \gamma^\mu A + (p'^\mu + p^\mu)B + (p'^\mu - p^\mu)C. \tag{5.2}$$

Further, A, B, and C are scalars and the only nontrivial scalar is q^2, which means that each of them must be functions of q^2, where $q = p' - p$ is the momentum of the photon. Multiplying the equation by q_μ, both the left hand side and the A and B terms vanish, but the C term does not (this is due to an identity known as Ward's identity, which will not be proved here). Thus, $C = 0$. We can also rewrite this expression using the Gordon identity, which says

$$\overline{u}(p')\gamma^\mu u(p) = \overline{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p), \tag{5.3}$$

where $\sigma^{\mu\nu}$ is the commutator (not anticommutator) of γ^μ and γ^ν. That is,

$$\sigma^{\mu\nu} = \frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right). \tag{5.4}$$

We use this to replace $p'^{\mu} + p^{\mu}$ with $i\sigma^{\mu\nu}q_{\nu}$. Thus we arrive at the expression for Γ^{μ},

$$\Gamma^{\mu}(p', p) = \gamma^{\mu}F_1(q^2) + \frac{i\sigma^{\mu\nu}}{2m}F_2(q^2), \tag{5.5}$$

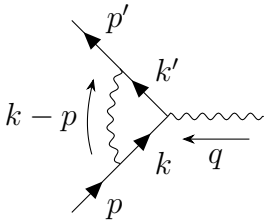
where we have written the coefficients as functions $F_1(q^2)$ and $F_2(q^2)$ known as *form factors*. Since Γ^{μ} encapsulates all possible interactions at the vertex, and Γ^{μ} is written in terms of F_1 and F_2, we expect the form factors to contain all information about how the electron responds to the electromagnetic field. In fact, it can be shown by using the Born approximation for scattering amplitudes that F_1 corresponds to the electric charge of the electron[5]. Similarly, the magnetic moment of the electron, $\boldsymbol{\mu}$, is given by

$$\boldsymbol{\mu} = g\left(\frac{e}{2m}\right)\mathbf{S}, \tag{5.6}$$

where $\mathbf{S}$ is the spin of the electron and g is the so called Landé g-factor, defined by

$$g = 2\left[F_1(0) + F_2(0)\right] = 2 + 2F_2(0). \tag{5.7}$$

To begin to evaluate the form factors, we consider only the left half of (5.1). That is,



$$= \overline{u}(p')\delta\Gamma^{\mu}u(p) \tag{5.8}$$

$$= \int \frac{d^4k}{(2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2 + i\epsilon}\overline{u}(p')(-ie\gamma^{\nu})\frac{i(\not{k}' + m)}{k'^2 - m^2 + i\epsilon}\gamma^{\mu}\frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}(-ie\gamma^{\rho})u(p).$$

Gathering only the numerator terms, using $g_{\nu\rho}$ to lower γ^{ρ} to γ^{ν}, and temporarily

omitting $\overline{u}(p')$, $u(p)$, and the overall factor of $-ie^2$, we see

$$\gamma^\nu(\not{k}' + m)\gamma^\mu(\not{k} + m)\gamma_\nu = \gamma^\nu \not{k}' \gamma^\mu \not{k} \gamma_\nu + \gamma^\nu \not{k}' \gamma^\mu m \gamma_\nu + \gamma^\nu m \gamma^\mu \not{k} \gamma_\nu + \gamma^\nu m \gamma^\mu m \gamma_\nu. \quad (5.9)$$

Putting this into a useful form will require a significant amount of algebra, but doing so now will help make a later substitution somewhat easier. One identity that needs to be established is $\gamma^\nu \gamma^\mu \gamma_\nu = -2\gamma^\mu$. This requires the anticommutation relations,

$$\begin{aligned} \gamma^\nu \gamma^\mu \gamma_\nu &= (2g^{\mu\nu} - \gamma^\mu \gamma^\nu)\gamma^\nu \\ &= 2g^{\mu\nu}\gamma_\nu - \gamma^\mu \gamma^\nu \gamma_\nu \\ &= 2\gamma^\mu - 4\gamma^\mu \\ &= -2\gamma^\mu. \end{aligned} \quad (5.10)$$

The general approach will be to manipulate each term until we can use this identity to eliminate γ^ν and γ_ν, and then to put the remaining terms in the same order, namely $\not{k}\gamma^\mu \not{k}'$. Doing so gives

$$\not{k}\gamma^\mu \not{k}' + m^2 \gamma^\mu - 2m(k + k')^\mu. \quad (5.11)$$

To get the denominator into a useful form will require the use of Feynman parameters. This is the process of introducing new variables x, y, and z (one for each factor in the denominator), and rewriting the expression using the identity

$$\frac{1}{ABC} = \int_0^1 dxdydz\delta(x + y + z - 1)\frac{2}{[Ax + By + Cz]^3}, \quad (5.12)$$

a proof of which can be found here [5]. Hence, we apply the formula to find

$$\frac{1}{((k-p)^2 + i\epsilon)(k'^2 - m^2 + i\epsilon)(k^2 - m^2 + i\epsilon)} = \int_0^1 dxdydz\delta(d + y + z - 1)\frac{2}{D^3}, \quad (5.13)$$

where

$$D = x(k^2 - m^2 + i\epsilon) + y(k'^2 - m^2 + i\epsilon) + z((k-p)^2 + i\epsilon). \tag{5.14}$$

Using the identities $k' = k + q$ and $x + y + z = 1$, one can show

$$D = k^2 + 2k \cdot (yq - zp) + yq^2 + zp^2 - (x+y)m^2 + i\epsilon. \tag{5.15}$$

We seek to complete the square, and so we introduce

$$\ell = k + yq - zp. \tag{5.16}$$

Notice that ℓ^2 is almost the same as D. Explicitly, making ample use of $x + y + z = 1$,

$$\begin{aligned} \ell^2 &= (k + yq - zp)(k + yq - zp) \\ &= D - xyq^2 - yzq^2 - xzp^2 - yzp^2 - 2yq \cdot p + 2xyq \cdot p + 2y^2 q \cdot p \\ &= D + \Delta, \end{aligned} \tag{5.17}$$

where we have defined Δ to absorb the "extra" terms so that

$$D = \ell^2 - \Delta + i\epsilon. \tag{5.18}$$

Again using $x + y + z = 1$, we can simplify the form for Δ,

$$\Delta = -xyq^2 + (1-z)^2 m^2. \tag{5.19}$$

Now we must express the numerator in terms of ℓ. To do so, we will make use of a couple of identities. The first is that

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{D^3} = 0. \tag{5.20}$$

This is true since it is an odd integrand integrated over symmetric limits. The other identity is that

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu \ell^\nu}{D^3} = \int \frac{d^4\ell}{(2\pi)^4}\frac{\frac{1}{4}g^{\mu\nu}\ell^2}{D^3}, \tag{5.21}$$

which can be verified by contracting both sides with $g_{\mu\nu}$.We can now substitute

$$k = \ell - yq + zp \tag{5.22}$$

and

$$k' = k + q = \ell + (1-y)q + zp \tag{5.23}$$

into (5.9). Then

$$\begin{aligned} \not{k}\gamma^\mu \not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu &= (\not{\ell} - y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu \\ &\quad - 2m(\ell^\mu - yq^\mu + zp^\mu + \ell^\mu + (1-y)q^\mu + zp^\mu) \\ &= -\frac{1}{2}\gamma^\mu\ell^2 + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p}) + m^2\gamma^\mu \\ &\quad - 2m((1-2y)q^\mu + 2zp^\mu). \end{aligned} \tag{5.24}$$

To see which parts of this expression correspond to which form factor F_1 or F_2, we must rewrite it in yet another form. Specifically, we seek to write it in the form

$$\gamma^\mu A + (p'^\mu + p^\mu)B + q^\mu C \tag{5.25}$$

just as in (5.2). To do so, we will make use of the Dirac equation so that $\not{p}u(p) = mu(p)$ and $\overline{u}(p')\not{p}' = \overline{u}(p')m$. Essentially, any time $\not{p}$ is the right-most factor in a given term, we may replace it with m, and similarly for $\not{p}'$ on the left. Using these, we also see that

$\overline{u}(p')\not{q}u(p) = 0$. After a significant amount of algebra, it can be shown that

$$\begin{aligned}(5.24) = \overline{u}(p')\Big[\gamma^\mu\Big(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + (1-2z-z^2)m^2\Big) \\ + (p'^\mu + p^\mu)mz(z-1) + q^\mu m(z-2)(x-y)\Big]u(p).\end{aligned} \tag{5.26}$$

The q^μ term vanishes upon integration since it is odd under switching x and y and is integrated over both. Using the Gordon identity from (5.3) and restoring the constant factor, the denominator, and the integrals, this becomes

$$\begin{aligned}\overline{u}(p')\Gamma^\mu(p',p)u(p) = \overline{u}(p')\Big[\gamma^\mu\Big(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\Big) \\ + \frac{i\sigma^{\mu\nu}q_\nu}{2m}2m^2z(1-z)\Big]u(p).\end{aligned} \tag{5.27}$$

Thus, we see that

$$\begin{aligned}F_1(q^2) = 2ie^2\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3} \\ \times\Big(-\frac{1}{2}\ell^2 + (1-x)(1-y)q^2 + (1-4z+z^2)m^2\Big)\end{aligned} \tag{5.28}$$

and

$$F_2(q^2) = 2ie^2\int\frac{d^4\ell}{(2\pi)^4}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3}2m^2z(1-z). \tag{5.29}$$

Evaluating F_1 is very difficult, as the integrals contain multiple divergences. As such, we will not attempt to evaluate it. The other form factor, however, does not contain any divergences and is much easier to compute. We first focus on the integral over ℓ. To make this integral easier to evaluate, we will want to convert to spherical coordinates. The barrier to doing so is that the integrand contains $D^3 = (\ell^2 - \Delta)^3$, but $\ell^2 = (\ell^0)^2 - |\boldsymbol{\ell}|^2$ is not the magnitude one uses in spherical coordinates. To fix this, we

perform what is called a *Wick rotation*. We define a new Euclidean variable ℓ_E by

$$\ell^0 = i\ell_E^0 \qquad \ell = \boldsymbol{\ell}_E. \tag{5.30}$$

Then

$$\begin{aligned} \ell^2 &= (i\ell_E^0)^2 - |\boldsymbol{\ell}|^2 \\ &= -(\ell_E^0)^2 - |\boldsymbol{\ell}_E|^2 \\ &= -\ell_E^2, \end{aligned} \tag{5.31}$$

where in the last line, ℓ_E denotes the Euclidean magnitude. Further, since $d\ell^0 = id\ell_E^0$, we have that $d^4\ell = id^4\ell_E$. Using a power of n for generality and using the fact that $\ell^2 = -\ell_E^2$, we see that

$$[\ell^2 - \Delta]^n = [-\ell_E^2 - \Delta]^n = (-1)^n(\ell_E^2 + \Delta)^n. \tag{5.32}$$

Plugging this in to (2.29) and omitting constants and variables without ℓ gives

$$\int \frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2 - \Delta]^n} = \frac{i}{(-1)^n}\frac{1}{2\pi^4}\int d^4\ell_E \frac{1}{[\ell_E^2 + \Delta^n]}. \tag{5.33}$$

This can now be converted to four-dimensional spherical coordinates to become

$$\frac{i(-1)^n}{(2\pi)^4}\int d\Omega_4 \int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2 + \Delta]^n}, \tag{5.34}$$

where $d^4\ell = \ell_E^3 d\Omega_4 d\ell_E$, and $\int d\Omega_4 = 2\pi^2$ is the "surface area" of the four-dimensional unit sphere. Ignoring the constants, we can evaluate

$$\int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2 + \Delta]^n} \tag{5.35}$$

by letting

$$\begin{aligned} u &= \ell_E^2 + \Delta & \ell_E^2 &= u - \Delta \\ du &= 2\ell_E d\ell_E & d\ell_E &= \frac{du}{2\ell_E} \end{aligned} \tag{5.36}$$

so that

$$\ell_E^3 d\ell_E = (u - \Delta)\frac{du}{2}. \tag{5.37}$$

We see that $u = \Delta$ when $\ell_E = 0$ and $u \to \infty$ as $\ell \to \infty$, so the integral becomes

$$\int_\Delta^\infty \frac{du}{2}\frac{u-\Delta}{u^n}. \tag{5.38}$$

Evaluating gives

$$\begin{aligned} \int_\Delta^\infty \frac{du}{2}\frac{u-\Delta}{u^n} &= \frac{1}{2}\int_\Delta^\infty du \frac{1}{u^{n-1}} - \frac{\Delta}{u^n} \\ &= \frac{1}{2}\Big[\frac{1}{2-n}u^{2-n} - \frac{\Delta}{1-n}u^{1-n}\Big]_\Delta^\infty \\ &= \frac{\Delta^{2-n}}{2(n-1)(n-2)}. \end{aligned} \tag{5.39}$$

Hence,

$$\frac{i(-1)^n}{(2\pi^2)}\int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2+\Delta]^n} = \frac{i(-1)^n}{(4\pi)^2}\frac{1}{(n-2)(n-1)\Delta^{n-2}}. \tag{5.40}$$

Letting $n = 3$ since (5.29) has a denominator of D^3, and plugging in $\Delta = m^2(1-z)^2 - xyq^2$ we obtain

$$F_2(q^2) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2m^2z(1-z)}{m^2(1-z)^2 - q^2xy}. \tag{5.41}$$

Since the correction to the magnetic moment from (5.7) depends on $F_2(0)$, that is what

we evaluate:

$$\begin{aligned} F_2(q^2=0) &= \frac{\alpha}{2\pi}\int_0^1 dxdydz\delta(x+y+z-1)\frac{2m^2z(1-z)}{m^2(1-z)^2} \\ &= \frac{\alpha}{\pi}\int_0^1 dz\int_0^{1-z} dy\frac{z}{1-z} \\ &= \frac{\alpha}{\pi}\int_0^1 dzz \\ &= \frac{\alpha}{2\pi}. \end{aligned} \tag{5.42}$$

We have finally found the correction to the magnetic moment of the electron, which agrees with experimental values.

Chapter 6

Applications

Theoretical physics, in particular QFT, provides an understanding of nature that has contributed to the development of several medical imaging techniques including PET scans, MRI scans, and HP gas MRI.

6.1 PET Scans

Recall the Dirac equation from (2.4),

$$(i\partial_\mu \gamma^\mu - m)\psi = 0, \tag{6.1}$$

where ψ is a four-component vector. It was found that this equation has some unusual solutions. Some solutions looked exactly as physicists expected, but in some solutions, the factor corresponding to energy had a negative sign. These "negative energy" solutions were at first deemed unphysical and extraneous and were ignored. Dirac, however, realized that this could be fixed by simply interpreting these solutions differently. He proposed that these solutions could describe a particle with the same mass as the electron, but with opposite charge: the positron. Since the theoretical positron has the opposite charge, it would indeed have equal and opposite energy as an

electron. Although the positron was first predicted in 1928, it was not actually discovered until 1932[3]. Since then, it has found a use in PET scans[6].

In positron emissions tomography scans, or PET scans, the patient is given a positron source such as ^{11}C or ^{15}O either by injection or ingestion. Once the positron source has sufficiently circulated throughout the body, a ring-shaped detector is placed around the area of interest. Similar to the process discussed in chapter 4, where an electron and a positron annihilate into a muon and an antimuon, there is the more probable process where an electron an a positron annihilate to become a pair of photons. Photons created in this way have a couple of particularly nice properties. The first is that each photon has a very specific energy, 511 keV, which is the rest mass of an electron. This is because of conservation of mass-energy. The initial electron and positron have a combined mass of 1022 keV and negligible kinetic energy. Thus, the final photon pair must have the same total amount of energy — 511 keV each. The second nice property is that the photons are emitted at 180° from each other, which allows the detector to easily calculate the point from which they were emitted. The detector collects data and uses it to produce an image of the tissue. PET scans are particularly useful for imaging cancerous tissue since the radioactive glucose tends to accumulate in tissue with high metabolic activity such as the brain or malignant tissue.

6.2 MRI scans

Magnetic resonance imaging scans, or MRI scans, use the magnetic moment of the proton to produce an image of tissue in the body[6]. The magnetic moment of a particle is a measure of how strongly the particle is affected by a magnetic field. More specifically, a proton can be in two possible states. It can either have spin that is parallel to the external magnetic field or it can have spin that is antiparallel to the external magnetic field. These two states have different energies where the difference is

given by

$$\Delta E = 2\mu_p B. \tag{6.2}$$

When a proton transitions from the state of higher energy (antiparallel) to the state of lower energy (parallel), it emits a photon with energy that is the same as the energy difference of the states. Since the energy of a photon is given by hf, where h is Planck's constant and f is the photon's frequency, we see that

$$\Delta E = 2\mu_p B = hf. \tag{6.3}$$

This frequency is known as the Larmor frequency.

The combined effect of the magnetic moment of many protons within tissue causes its own magnetic field. As the protons switch from one state to the other, there is an overall change in the magnetic field which can be measured and used to produce the image.

Clearly, the magnetic moment is crucial to this process. In Chapter 5, we used QFT to make a correction to the predicted value of the magnetic moment of the electron. Unfortunately, it is not quite as simple to use QFT to make similar predictions about the magnetic moment of the proton. This is because the proton is a composite particle; it is composed of three smaller particles known as quarks. To make predictions about the proton would require having an accurate model for the proton as a bound state of quarks. Finding this model, known as the "parton structure of the proton" is not easy to do and is an active area of research[7].

Both PET and MRI have some disadvantages. PET scans offer lower spatial resolution and they also use ionizing radiation which can increase the risk of complications due to radiation exposure. MRI requires a much higher concentration of metabolites to produce a clear image. Thus, there have been recent developments of a

scanner that can image both PET and MRI scans simultaneously. Although this can allow for each type of imaging to complement the deficiencies of the other, this method also introduces several challenges. One such challenge is that the magnetic field used for MRI can interfere with the detectors used for PET, reducing the quality of the image. Another challenge is that the apparatus must have a larger radius than a typical MRI to make room for the ring of detectors for PET. This can significantly increase the cost and power consumption since these both greatly increase with the radius of the MRI chamber[8]. Although MRI does not use ionizing radiation, recent studies posit that MRI may increase the long-term risk of DNA damage, but this is unconfirmed and requires further research[9].

6.3 HP gas MRI

A more recently developed technique related to the one just discussed is hyperpolarized gas magnetic resonance imaging, or HP gas MRI[10]. As the name suggests, the general principle is similar to that of MRI in that it relies on measuring the spin states of a particle within the body. However, conventional MRI uses the protons in hydrogen atoms, whereas HP gas MRI uses electrons in an inert gas such as ^{3}He or ^{123}Xe. HP gas MRI was originally tested and developed using ^{3}He, however, ^{3}He is difficult to produce and is thus more expensive, making ^{123}Xe the new preference. HP gas MRI is particularly well suited to imaging the lungs. Since conventional MRI relies on water content, it cannot image the lungs which are mostly gas. However, HP gas MRI comes with its own challenges, the primary of which is that the density of gas in the lungs is about 1/1000th the density of water in the rest of the body and thus, the signal is much weaker. This is remedied by hyperpolarizing the xenon gas by artificially causing more electrons than normal to be in a given spin state so that there is a greater rate of transitions and thus, a stronger signal. The xenon gas used in MRI has not been

found to have any adverse effects[10].

There is a similar form of hyperpolarized MRI which does not use a gas, but instead uses ^{13}C. This method uses metabolites that have been marked with hyperpolarized ^{13}C to image specific metabolic pathways. Although HP MRI is a promising new medical imaging technique, the cost and access are currently prohibitive. Recently published schematics for an open source ^{123}Xe polarizer system may help to improve access in the near future[11].

Just as conventional MRI depends on the magnetic moment of the proton, HP gas MRI depends on the magnetic moment of the electron to which we were able to successfully calculate a correction in Chapter 6, which is in agreement with experiment.

Bibliography

[1]K. Krane, *Modern physics*, 2nd ed. (Wiley, 1996).

[2]R. Fitzpatrick, *Quantum mechanics* (World Scientific Publishing, 2015).

[3]P. A. M. Dirac, Proceedings of the Royal Society **117**, 610 (1928).

[4]J. Taylor, *Classical mechanics* (University Science Books, 2005).

[5]M. Peskin and D. Schroeder, *Quantum field theory* (Cambridge University Press, 2007).

[6]J. Lilley, *Nuclear physics* (Wiley India, Daryaganj, 2007), p. 246.

[7]J. Karpie, K. Orginos, A. Rothkopf, and S. Zafeiropoulos, Journal of High Energy Physics **2019**, 57 (2019).

[8]C. Catana, A. R. Guimaraes, and B. R. Rosen, Journal of Nuclear Medicine **54**, 815 (2013).

[9]M. A. Hill, P. O'Neill, and W. G. McKenna, European Heart Journal - Cardiovascular Imaging **17**, 1230 (2016).

[10]J. E. Roos, H. P. McAdams, S. S. Kaushik, and B. Driehuys, Magnetic Resonance Imaging Clinics of North America **23**, 217 (2015).

[11]E. B. Adamson, K. D. Ludwig, D. G. Mummy, and S. B. Fain, Physics in Medicine and Biology **62**, R81 (2017).

[12]M. Srednicki, *Quantum field theory* (Cambridge University Press, 2007).

[13]S. Weinberg, *Quantum theory of fields*, Vol. 1 (Cambridge University Press, 1995).

Vita

C.J. Brewer was born and raised in San Angelo, Texas. He attended Grape Creek ISD from pre-K through 12th grade, where he graduated as valedictorian of his high school class. He graduated from Angelo State University with a Bachelor of Science in Mathematics and Physics with Highest University Honors as the first physics student to complete a thesis. He will attend the Mathematics Ph.D. program at the University of Texas at Dallas, where he plans to pursue research in mathematical physics.